

Field theory on nonanticommutative superspace

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ABSTRACT: We discuss a deformation of the Hopf algebra of supersymmetry (SUSY) transformations based on the special choice of twist. As usual, algebra itself remains unchanged, but the comultiplication changes. This leads to the deformed Leibniz rule for SUSY transformations. Superfields are elements of the algebra of functions of the usual supercoordinates. Elements of this algebra are multiplied by using the \star -product which is noncommutative, hermitian and finite when expanded in power series of the deformation parameter. Chiral fields are no longer a subalgebra of the algebra of superfields. One possible deformation of the Wess-Zumino action is proposed and analyzed in detail. Differently from most of the literature concerning this subject, we work in Minkowski space-time.

KEYWORDS: Superspaces, Non-Commutative Geometry.

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1. Introduction

It is well known that Quantum Field Theory (QFT) encounters problems at very high energies and very short distances. This suggests that the structure of space-time has to be modified at these scales. One possibility to modify the structure of space-time is to deform the usual commutation relations between coordinates; this gives a noncommutative (NC) space [1]. Different models of noncommutativity were discussed in the literature. One of the simplest examples is the θ -deformed or canonically deformed space-time [2] with

$$[x^m, x^n] = i\theta^{mn}. \tag{1.1}$$

Here θ^{mn} is a constant antisymmetric matrix. Gauge theories were defined and analyzed in details in this framework [3]. Also, a deformed Standard Model was formulated [4] and renormalizability properties of field theories on this space are subject of many papers [5].

More complicated deformations of space-time, such as κ -deformation [6] and q -deformation [7] were also discussed in the literature.

In order to understand the physics at very small scales better, in recent years attempts were made to combine supersymmetry with noncommutativity. In [8] the authors combine SUSY with the κ -deformation of space-time, while in [9] SUSY is combined with the canonical deformation of space-time. In series of papers [10–12] a version of non(anti)commutative superspace is defined and analyzed. The anticommutation relations between the fermionic coordinates are modified in the following way

$$\{\theta^\alpha \star \theta^\beta\} = C^{\alpha\beta}, \quad \{\bar{\theta}_{\dot{\alpha}} \star \bar{\theta}_{\dot{\beta}}\} = \{\theta^\alpha \star \bar{\theta}_{\dot{\alpha}}\} = 0, \tag{1.2}$$

where $C^{\alpha\beta} = C^{\beta\alpha}$ is a complex, constant symmetric matrix. Such deformation is well defined only in Euclidean space where undotted and dotted spinors are not related by the usual complex conjugation. Note that the chiral coordinates $y^m = x^m + i\theta\sigma^m\bar{\theta}$ commute in this setting.

In [11] the notion of chirality is preserved, i.e. the deformed product of two chiral superfields is again a chiral superfield. On the other hand, one half of $N = 1$ supersymmetry is broken and this is the so-called $N = 1/2$ supersymmetry. Another type of deformation is introduced in [12]. There the product of two chiral superfields is not a chiral superfield but the model is invariant under the full supersymmetry. The Hopf algebra of SUSY transformations is deformed by using the twist approach in [13]. Examples of deformation that introduce nontrivial commutation relations between chiral and fermionic coordinates are discussed in [14]. Some consequences of nontrivial (anti)commutation relations on statistics and S-matrix are analyzed in [15].

In this paper we apply a twist to deform the Hopf algebra of SUSY transformations. However, our choice of the twist is different from that in [13] since we want to work in Minkowski space-time. As undotted and dotted spinors are related by the usual complex conjugation, we obtain

$$\{\theta^\alpha \star \theta^\beta\} = C^{\alpha\beta}, \quad \{\bar{\theta}_{\dot{\alpha}} \star \bar{\theta}_{\dot{\beta}}\} = \bar{C}_{\dot{\alpha}\dot{\beta}}, \quad \{\theta^\alpha \star \bar{\theta}_{\dot{\alpha}}\} = 0, \quad (1.3)$$

with $\bar{C}_{\dot{\alpha}\dot{\beta}} = (C_{\alpha\beta})^*$. Our main goal is the formulation and analysis of the deformed Wess-Zumino Lagrangian.

The paper is organized as follows: In section 2 we review the undeformed supersymmetric theory to establish the notation and then rewrite it by using the language of Hopf algebras. We follow the notation of [16]. By twisting the Hopf algebra of SUSY transformations, a Hopf algebra of deformed SUSY transformations is obtained in section 3. As the algebra itself remains undeformed, the full $N = 1$ SUSY is preserved. On the other hand, the comultiplication changes and that leads to a deformed Leibniz rule. As a consequence of the twist, a \star -product is introduced on the algebra of functions of supercoordinates. Sections 4 and 5 are devoted to the construction of a deformed Wess-Zumino Lagrangian. Since our choice of the twist implies that the \star -product of chiral superfields is not a chiral superfield we have to use (anti)chiral projectors to project irreducible components of such \star -products. In the section 6 the auxiliary fields are integrated out and the expansion in the deformation parameter of the "on-shell" action is given. Some consequences of applying the twist on the Poincaré invariance are discussed in the section 7. Two examples of how to apply the deformed Leibniz rule when transforming \star -products of fields are given. Finally, we end the paper with some short comments and conclusions.

2. Undeformed SUSY transformations

The undeformed superspace is generated by x , θ and $\bar{\theta}$ coordinates which fulfill

$$\begin{aligned} [x^m, x^n] &= [x^m, \theta^\alpha] = [x^m, \bar{\theta}_{\dot{\alpha}}] = 0, \\ \{\theta^\alpha, \theta^\beta\} &= \{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = \{\theta^\alpha, \bar{\theta}_{\dot{\alpha}}\} = 0, \end{aligned} \quad (2.1)$$

with $m = 0, \dots, 3$ and $\alpha, \beta = 1, 2$. These coordinates we call the supercoordinates, to x^m we refer as to bosonic and to θ^α and $\bar{\theta}_{\dot{\alpha}}$ we refer as to fermionic coordinates. Also, $x^2 = x^m x_m = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$, that is we work in Minkowski space-time with the metric $(-, +, +, +)$.

Every function of the supercoordinates can be expanded in power series in θ and $\bar{\theta}$. Superfields form a subalgebra of the algebra of functions on the superspace. For a general superfield $F(x, \theta, \bar{\theta})$ the expansion in θ and $\bar{\theta}$ reads

$$F(x, \theta, \bar{\theta}) = f(x) + \theta\phi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta}n(x) + \theta\sigma^m\bar{\theta}v_m + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\varphi(x) + \theta\theta\bar{\theta}\bar{\theta}d(x). \quad (2.2)$$

All higher powers of θ and $\bar{\theta}$ vanish since these coordinates are Grassmanian.

Under the infinitesimal SUSY transformations a general superfield transforms as

$$\delta_\xi F = (\xi Q + \bar{\xi}\bar{Q})F, \quad (2.3)$$

where ξ and $\bar{\xi}$ are constant anticommuting parameters and Q and \bar{Q} are SUSY generators

$$Q_\alpha = \partial_\alpha - i\sigma_{\alpha\dot{\alpha}}^m \bar{\theta}^{\dot{\alpha}} \partial_m, \quad (2.4)$$

$$\bar{Q}^{\dot{\alpha}} = \bar{\partial}^{\dot{\alpha}} - i\theta^\alpha \sigma_{\alpha\dot{\beta}}^m \varepsilon^{\dot{\beta}\dot{\alpha}} \partial_m. \quad (2.5)$$

Using the expansion (2.2) one can calculate the transformation law of the component fields

$$\delta_\xi f = \xi^\alpha \phi_\alpha + \bar{\xi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}, \quad (2.6)$$

$$\delta_\xi \phi_\alpha = 2\xi_\alpha m + \sigma_{\alpha\dot{\alpha}}^m \bar{\xi}^{\dot{\alpha}} (v_m + i(\partial_m f)), \quad (2.7)$$

$$\delta_\xi \bar{\chi}^{\dot{\alpha}} = 2\bar{\xi}^{\dot{\alpha}} n + \bar{\sigma}^{m\dot{\alpha}\alpha} \xi_\alpha (-v_m + i(\partial_m f)), \quad (2.8)$$

$$\delta_\xi m = \bar{\xi}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} + \frac{i}{2} \bar{\xi}_{\dot{\alpha}} \bar{\sigma}^{m\dot{\alpha}\alpha} (\partial_m \phi_\alpha), \quad (2.9)$$

$$\delta_\xi n = \xi^\alpha \varphi_\alpha + \frac{i}{2} \xi^\alpha \sigma_{\alpha\dot{\alpha}}^m (\partial_m \bar{\chi}^{\dot{\alpha}}), \quad (2.10)$$

$$\sigma_{\alpha\dot{\alpha}}^m \delta_\xi v_m = -i(\partial_m \phi_\alpha) \xi^\beta \sigma_{\beta\dot{\alpha}}^m + 2\xi_\alpha \bar{\lambda}^{\dot{\alpha}} + i\sigma_{\alpha\dot{\beta}}^m \bar{\xi}^{\dot{\beta}} (\partial_m \bar{\chi}^{\dot{\alpha}}) + 2\varphi_\alpha \bar{\xi}^{\dot{\alpha}}, \quad (2.11)$$

$$\delta_\xi \bar{\lambda}^{\dot{\alpha}} = 2\bar{\xi}^{\dot{\alpha}} d + i\bar{\sigma}^{l\dot{\alpha}\alpha} \xi_\alpha (\partial_l m) + \frac{i}{2} \bar{\sigma}^{l\dot{\alpha}\alpha} \sigma_{\alpha\dot{\beta}}^m \bar{\xi}^{\dot{\beta}} (\partial_m v_l), \quad (2.12)$$

$$\delta_\xi \varphi_\alpha = 2\xi_\alpha d + i\sigma_{\alpha\dot{\alpha}}^l \bar{\xi}^{\dot{\alpha}} (\partial_l n) - \frac{i}{2} \sigma_{\alpha\dot{\alpha}}^l \bar{\sigma}^{m\dot{\alpha}\beta} \xi_\beta (\partial_m v_l), \quad (2.13)$$

$$\delta_\xi d = \frac{i}{2} \xi^\alpha \sigma_{\alpha\dot{\alpha}}^m (\partial_m \bar{\lambda}^{\dot{\alpha}}) - \frac{i}{2} (\partial_m \varphi^\alpha) \sigma_{\alpha\dot{\alpha}}^m \bar{\xi}^{\dot{\alpha}}. \quad (2.14)$$

Transformations (2.3) close in the algebra

$$[\delta_\xi, \delta_\eta] = -2i(\eta\sigma^m\bar{\xi} - \xi\sigma^m\bar{\eta})\partial_m. \quad (2.15)$$

We next consider the product of two superfields defined as

$$F \cdot G = \mu\{F \otimes G\}, \quad (2.16)$$

where the bilinear map μ maps the tensor product to the space of functions. The transformation law of the product (2.16) is given by

$$\begin{aligned}\delta_\xi(F \cdot G) &= (\xi Q + \bar{\xi} \bar{Q})(F \cdot G), \\ &= (\delta_\xi F) \cdot G + F \cdot (\delta_\xi G).\end{aligned}\tag{2.17}$$

The first line tells us that the product of two superfields is a superfield again. The second line is the usual Leibniz rule.

All these properties we summarise in the language of Hopf algebras [7], which will be useful when we introduce a deformation of the superspace. The Hopf algebra of undeformed SUSY transformations is given by

- algebra

$$[\delta_\xi, \delta_\eta] = -2i(\eta\sigma^m\bar{\xi} - \xi\sigma^m\bar{\eta})\partial_m, \quad [\partial_m, \partial_n] = [\partial_m, \delta_\xi] = 0.$$

- coproduct

$$\Delta(\delta_\xi) = \delta_\xi \otimes 1 + 1 \otimes \delta_\xi, \quad \Delta\partial_m = \partial_m \otimes 1 + 1 \otimes \partial_m.\tag{2.18}$$

- counit and antipode

$$\varepsilon(\delta_\xi) = \varepsilon(\partial_m) = 0, \quad S(\delta_\xi) = -\delta_\xi, \quad S(\partial_m) = -\partial_m.\tag{2.19}$$

In the language of generators Q_α and $\bar{Q}_{\dot{\alpha}}$ this Hopf algebra reads

- algebra

$$\begin{aligned}\{Q_\alpha, Q_\beta\} &= \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, & \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} &= 2i\sigma_{\alpha\dot{\beta}}^m\partial_m, \\ [\partial_m, \partial_n] &= [\partial_m, Q_\alpha] = [\partial_m, \bar{Q}_{\dot{\alpha}}] = 0.\end{aligned}\tag{2.20}$$

- coproduct

$$\begin{aligned}\Delta Q_\alpha &= Q_\alpha \otimes 1 + 1 \otimes Q_\alpha, & \Delta \bar{Q}_{\dot{\alpha}} &= \bar{Q}_{\dot{\alpha}} \otimes 1 + 1 \otimes \bar{Q}_{\dot{\alpha}}, \\ \Delta \partial_m &= \partial_m \otimes 1 + 1 \otimes \partial_m.\end{aligned}\tag{2.21}$$

- counit and antipode

$$\begin{aligned}\varepsilon(Q_\alpha) &= \varepsilon(\bar{Q}_{\dot{\alpha}}) = \varepsilon(\partial_m) = 0, \\ S(Q_\alpha) &= -Q_\alpha, \quad S(\bar{Q}_{\dot{\alpha}}) = -\bar{Q}_{\dot{\alpha}}, \quad S(\partial_m) = -\partial_m.\end{aligned}\tag{2.22}$$

3. Twisted SUSY transformations

As in [17] we introduce the deformed SUSY transformations by twisting the usual Hopf algebra (2.18). For the twist \mathcal{F} we choose

$$\mathcal{F} = e^{\frac{1}{2}C^{\alpha\beta}\partial_\alpha\otimes\partial_\beta + \frac{1}{2}\bar{C}_{\dot{\alpha}\dot{\beta}}\bar{\partial}^{\dot{\alpha}}\otimes\bar{\partial}^{\dot{\beta}}}, \quad (3.1)$$

with $C^{\alpha\beta} = C^{\beta\alpha}$ a complex constant matrix. Note that $C^{\alpha\beta}$ and $\bar{C}^{\dot{\alpha}\dot{\beta}}$ are related by the usual complex conjugation. It was shown in [18] that (3.1) satisfies all the requirements for a twist [19]. The twisted Hopf algebra of SUSY transformation now reads

- algebra

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, & \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} &= 2i\sigma_{\alpha\dot{\beta}}^m\partial_m, \\ [\partial_m, \partial_n] &= [\partial_m, \partial_\alpha] = [\partial_m, \bar{\partial}_{\dot{\beta}}] = [\partial_m, Q_\alpha] = [\partial_m, \bar{Q}_{\dot{\alpha}}] = 0, \\ \{\partial_\alpha, \partial_\beta\} &= \{\partial_\alpha, \bar{\partial}_{\dot{\beta}}\} = \{\bar{\partial}_{\dot{\alpha}}, \bar{\partial}_{\dot{\beta}}\} = \{\partial_\alpha, Q_\beta\} = \{\bar{\partial}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, \\ \{\partial_\alpha, \bar{Q}_{\dot{\alpha}}\} &= -i\sigma_{\alpha\dot{\beta}}^m\varepsilon^{\dot{\beta}\dot{\alpha}}\partial_m, & \{\bar{\partial}_{\dot{\alpha}}, Q_\alpha\} &= -i\sigma_{\alpha\dot{\alpha}}^m\partial_m. \end{aligned} \quad (3.2)$$

- coproduct

$$\begin{aligned} \Delta_{\mathcal{F}}(Q_\alpha) &= \mathcal{F}(Q_\alpha \otimes 1 + 1 \otimes Q_\alpha)\mathcal{F}^{-1} \\ &= Q_\alpha \otimes 1 + 1 \otimes Q_\alpha \\ &\quad - \frac{i}{2}\bar{C}_{\dot{\alpha}\dot{\beta}}\left(\sigma_{\alpha\dot{\gamma}}^m\varepsilon^{\dot{\gamma}\dot{\alpha}}\partial_m \otimes \bar{\partial}^{\dot{\beta}} + \bar{\partial}^{\dot{\alpha}} \otimes \sigma_{\alpha\dot{\gamma}}^m\varepsilon^{\dot{\gamma}\dot{\beta}}\partial_m\right), \\ \Delta_{\mathcal{F}}(\bar{Q}_{\dot{\alpha}}) &= \bar{Q}_{\dot{\alpha}} \otimes 1 + 1 \otimes \bar{Q}_{\dot{\alpha}} \\ &\quad + \frac{i}{2}C^{\alpha\beta}\left(\sigma_{\alpha\dot{\alpha}}^m\partial_m \otimes \partial_\beta + \partial_\alpha \otimes \sigma_{\beta\dot{\alpha}}^m\partial_m\right), \\ \Delta\partial_m &= \partial_m \otimes 1 + 1 \otimes \partial_m, \\ \Delta\partial_\alpha &= \partial_\alpha \otimes 1 + 1 \otimes \partial_\alpha, & \Delta\bar{\partial}^{\dot{\alpha}} &= \bar{\partial}^{\dot{\alpha}} \otimes 1 + 1 \otimes \bar{\partial}^{\dot{\alpha}}. \end{aligned} \quad (3.3)$$

- counit and antipode

$$\begin{aligned} \varepsilon(Q_\alpha) &= \varepsilon(\bar{Q}_{\dot{\alpha}}) = \varepsilon(\partial_m) = \varepsilon(\partial_\alpha) = \varepsilon(\bar{\partial}^{\dot{\alpha}}) = 0, \\ S(Q_\alpha) &= -Q_\alpha, & S(\bar{Q}_{\dot{\alpha}}) &= -\bar{Q}_{\dot{\alpha}}, \\ S(\partial_m) &= -\partial_m, & S(\partial_\alpha) &= -\partial_\alpha, & S(\bar{\partial}^{\dot{\alpha}}) &= -\bar{\partial}^{\dot{\alpha}}. \end{aligned} \quad (3.4)$$

Note that only the coproduct is changed, while the algebra stays the same as in the undeformed case. This means that the full supersymmetry is preserved. Also note that in order for the comultiplication for Q_α and $\bar{Q}_{\dot{\alpha}}$ to close in the algebra, we had to enlarge the algebra by introducing the fermionic derivatives ∂_α and $\bar{\partial}_{\dot{\alpha}}$.

The inverse of the twist (3.1)

$$\mathcal{F}^{-1} = e^{-\frac{1}{2}C^{\alpha\beta}\partial_\alpha\otimes\partial_\beta - \frac{1}{2}\bar{C}_{\dot{\alpha}\dot{\beta}}\bar{\partial}^{\dot{\alpha}}\otimes\bar{\partial}^{\dot{\beta}}}, \quad (3.5)$$

defines a new product on the algebra of functions of supercoordinates called the \star -product. For two arbitrary superfields F and G the \star -product is defined as follows

$$\begin{aligned}
 F \star G &= \mu_\star \{F \otimes G\} \\
 &= \mu \{ \mathcal{F}^{-1} F \otimes G \} \\
 &= \mu \{ e^{-\frac{1}{2} C^{\alpha\beta} \partial_\alpha \otimes \partial_\beta - \frac{1}{2} \bar{C}_{\dot{\alpha}\dot{\beta}} \bar{\partial}^{\dot{\alpha}} \otimes \bar{\partial}^{\dot{\beta}}} F \otimes G \} \\
 &= F \cdot G - \frac{1}{2} (-1)^{|F|} C^{\alpha\beta} (\partial_\alpha F) \cdot (\partial_\beta G) - \frac{1}{2} (-1)^{|F|} \bar{C}_{\dot{\alpha}\dot{\beta}} (\bar{\partial}^{\dot{\alpha}} F) (\bar{\partial}^{\dot{\beta}} G) \\
 &\quad - \frac{1}{8} C^{\alpha\beta} C^{\gamma\delta} (\partial_\alpha \partial_\gamma F) \cdot (\partial_\beta \partial_\delta G) - \frac{1}{8} \bar{C}_{\dot{\alpha}\dot{\beta}} \bar{C}_{\dot{\gamma}\dot{\delta}} (\bar{\partial}^{\dot{\alpha}} \bar{\partial}^{\dot{\gamma}} F) (\bar{\partial}^{\dot{\beta}} \bar{\partial}^{\dot{\delta}} G) \\
 &\quad - \frac{1}{4} C^{\alpha\beta} \bar{C}_{\dot{\alpha}\dot{\beta}} (\partial_\alpha \bar{\partial}^{\dot{\alpha}} F) (\partial_\beta \bar{\partial}^{\dot{\beta}} G) \\
 &\quad + \frac{1}{16} (-1)^{|F|} C^{\alpha\beta} C^{\gamma\delta} \bar{C}_{\dot{\alpha}\dot{\beta}} (\partial_\alpha \partial_\gamma \bar{\partial}^{\dot{\alpha}} F) (\partial_\beta \partial_\delta \bar{\partial}^{\dot{\beta}} G) \\
 &\quad + \frac{1}{16} (-1)^{|F|} C^{\alpha\beta} \bar{C}_{\dot{\alpha}\dot{\beta}} \bar{C}_{\dot{\gamma}\dot{\delta}} (\partial_\alpha \bar{\partial}^{\dot{\alpha}} \bar{\partial}^{\dot{\gamma}} F) (\partial_\beta \bar{\partial}^{\dot{\beta}} \bar{\partial}^{\dot{\delta}} G) \\
 &\quad + \frac{1}{64} C^{\alpha\beta} C^{\gamma\delta} \bar{C}_{\dot{\alpha}\dot{\beta}} \bar{C}_{\dot{\gamma}\dot{\delta}} (\partial_\alpha \partial_\gamma \bar{\partial}^{\dot{\alpha}} \bar{\partial}^{\dot{\gamma}} F) (\partial_\beta \partial_\delta \bar{\partial}^{\dot{\beta}} \bar{\partial}^{\dot{\delta}} G),
 \end{aligned} \tag{3.6}$$

where $|F| = 1$ if F is odd (fermionic) and $|F| = 0$ if F is even (bosonic). In the second line the definition of the multiplication μ_\star is given. No higher powers of $C^{\alpha\beta}$ and $\bar{C}_{\dot{\alpha}\dot{\beta}}$ appear since the derivatives ∂_α and $\bar{\partial}^{\dot{\alpha}}$ are Grassmanian. Expansion of the \star -product (3.7) ends after the 4th order in the deformation parameter. This is different from the case of the Moyal-Weyl \star_{mw} -product [2, 20] where the expansion in powers of the deformation parameter leads to an infinite power series. One should also note that the \star -product (3.7) is hermitian,

$$(F \star G)^* = G^* \star F^*, \tag{3.8}$$

where $*$ denotes the usual complex conjugation. This is important for the construction of physical models.

The \star -product (3.7) gives

$$\begin{aligned}
 \{\theta^\alpha \star \theta^\beta\} &= C^{\alpha\beta}, & \{\bar{\theta}_{\dot{\alpha}} \star \bar{\theta}_{\dot{\beta}}\} &= \bar{C}_{\dot{\alpha}\dot{\beta}}, & \{\theta^\alpha \star \bar{\theta}_{\dot{\alpha}}\} &= 0, \\
 [x^m \star x^n] &= 0, & [x^m \star \theta^\alpha] &= 0, & [x^m \star \bar{\theta}_{\dot{\alpha}}] &= 0.
 \end{aligned} \tag{3.9}$$

Note that the chiral coordinates y^m do not commute in this setting, but instead fulfill

$$\begin{aligned}
 [y^m \star y^n] &= -\theta \theta \bar{C}^{\dot{\alpha}\dot{\beta}} \varepsilon_{\dot{\beta}\dot{\gamma}} (\bar{\sigma}^{mn})^{\dot{\gamma}}_{\dot{\alpha}} - \bar{\theta} \bar{\theta} \varepsilon_{\alpha\beta} C^{\beta\gamma} (\sigma^{mn})_{\gamma}^{\alpha}, \\
 [y^m \star \theta^\alpha] &= i C^{\alpha\beta} \sigma_{\beta\dot{\beta}}^m \bar{\theta}^{\dot{\beta}}, & [y^m \star \bar{\theta}_{\dot{\alpha}}] &= i \theta^\alpha \sigma_{\alpha\dot{\beta}}^m \bar{C}^{\dot{\beta}\dot{\alpha}}.
 \end{aligned} \tag{3.10}$$

Relations (3.9) enable us to define the deformed superspace or "nonanticommutative space". It is generated by the usual bosonic and fermionic coordinates (2.1) while the deformation is contained in the new product (3.7).

The deformed infinitesimal SUSY transformation is defined in the following way

$$\begin{aligned}
 \delta_\xi^\star F &= (\xi Q + \bar{\xi} \bar{Q}) F \\
 &= X_{\xi Q}^\star \star F + X_{\bar{\xi} \bar{Q}}^\star \star F.
 \end{aligned} \tag{3.11}$$

Differential operators $X_{\xi Q}^*$ and $X_{\bar{\xi} \bar{Q}}^*$ are given by

$$\begin{aligned} X_{\xi Q}^* &= \xi^\alpha \left(Q_\alpha + \frac{1}{2} \bar{C}_{\dot{\beta}\dot{\gamma}} (\bar{\partial}^{\dot{\beta}} Q_\alpha) \bar{\partial}^{\dot{\gamma}} \right) \\ &= \xi^\alpha \left(Q_\alpha + \frac{i}{2} \bar{C}_{\dot{\beta}\dot{\gamma}} \sigma_{\alpha\dot{\alpha}}^m \varepsilon^{\dot{\alpha}\dot{\beta}} \partial_m \bar{\partial}^{\dot{\gamma}} \right), \end{aligned} \quad (3.12)$$

$$\begin{aligned} X_{\bar{\xi} \bar{Q}}^* &= \bar{\xi}_{\dot{\alpha}} \left(\bar{Q}^{\dot{\alpha}} + \frac{1}{2} C^{\alpha\beta} (\partial_\alpha \bar{Q}^{\dot{\alpha}}) \partial_\beta \right) \\ &= \bar{\xi}_{\dot{\alpha}} \left(\bar{Q}^{\dot{\alpha}} - \frac{i}{2} C^{\alpha\beta} \sigma_{\alpha\dot{\gamma}}^m \partial_m \partial_\beta \right). \end{aligned} \quad (3.13)$$

Note that X^* operators close in the following algebra

$$\{X_{Q_\alpha}^* \star X_{Q_\beta}^*\} = \{X_{\bar{Q}^{\dot{\alpha}}}^* \star X_{\bar{Q}^{\dot{\beta}}}^*\} = 0, \quad \{X_{Q_\alpha}^* \star X_{\bar{Q}^{\dot{\beta}}}^*\} = 2i\sigma_{\alpha\dot{\alpha}}^m \partial_m. \quad (3.14)$$

This is just a different way of writing the algebra (3.2). Differential operators X^* are mentioned in [11], however no detailed analysis is performed. In [21] the authors discuss the Supersymmetric Quantum Mechanics with odd-parameters being Clifford-valued and the operators similar to (3.12) and (3.13) arise.

The deformed coproduct (3.3) insures that the \star -product of two superfields is again a superfield. Its transformation law is given by

$$\begin{aligned} \delta_\xi^*(F \star G) &= (\xi Q + \bar{\xi} \bar{Q})(F \star G), \\ &= \mu_\star \{ \Delta_{\mathcal{F}}(\delta_\xi^*) F \otimes G \}, \end{aligned} \quad (3.15)$$

with

$$\begin{aligned} \Delta_{\mathcal{F}}(\delta_\xi^*) &= \mathcal{F} \left(\delta_\xi^* \otimes 1 + 1 \otimes \delta_\xi^* \right) \mathcal{F}^{-1} \\ &= \delta_\xi^* \otimes 1 + 1 \otimes \delta_\xi^* + \frac{i}{2} C^{\alpha\beta} \left(\bar{\xi}^{\dot{\gamma}} \sigma_{\alpha\dot{\gamma}}^m \partial_m \otimes \partial_\beta + \partial_\beta \otimes \bar{\xi}^{\dot{\gamma}} \sigma_{\alpha\dot{\gamma}}^m \partial_m \right) \\ &\quad - \frac{i}{2} \bar{C}_{\dot{\alpha}\dot{\beta}} \left(\xi^\alpha \sigma_{\alpha\dot{\gamma}}^m \varepsilon^{\dot{\gamma}\dot{\alpha}} \partial_m \otimes \bar{\partial}^{\dot{\beta}} + \bar{\partial}^{\dot{\alpha}} \otimes \xi^\alpha \sigma_{\alpha\dot{\gamma}}^m \varepsilon^{\dot{\gamma}\dot{\beta}} \partial_m \right). \end{aligned}$$

This gives

$$\begin{aligned} \delta_\xi^*(F \star G) &= (\delta_\xi^* F) \star G + F \star (\delta_\xi^* G) \\ &\quad + \frac{i}{2} C^{\alpha\beta} \left(\bar{\xi}^{\dot{\gamma}} \sigma_{\alpha\dot{\gamma}}^m (\partial_m F) \star (\partial_\beta G) + (\partial_\alpha F) \star \bar{\xi}^{\dot{\gamma}} \sigma_{\beta\dot{\gamma}}^m (\partial_m G) \right) \\ &\quad - \frac{i}{2} \bar{C}_{\dot{\alpha}\dot{\beta}} \left(\xi^\alpha \sigma_{\alpha\dot{\gamma}}^m \varepsilon^{\dot{\gamma}\dot{\alpha}} (\partial_m F) \star (\bar{\partial}^{\dot{\beta}} G) + (\bar{\partial}^{\dot{\alpha}} F) \star \xi^\alpha \sigma_{\alpha\dot{\gamma}}^m \varepsilon^{\dot{\gamma}\dot{\beta}} (\partial_m G) \right). \end{aligned} \quad (3.16)$$

4. Chiral fields

Having established the general properties of the introduced deformation we now turn to one special example, namely we study chiral fields. In the undeformed theory chiral fields form a subalgebra of the algebra of superfields. In the deformed case this will no longer be the case.

A chiral field Φ fulfills $\bar{D}_{\dot{\alpha}}\Phi = 0$, where $\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^{\alpha}\sigma_{\alpha\dot{\alpha}}^m\partial_m$ is the supercovariant derivative. In terms of component fields the chiral superfield Φ is given by

$$\begin{aligned}\Phi(x, \theta, \bar{\theta}) &= A(x) + \sqrt{2}\theta^{\alpha}\psi_{\alpha}(x) + \theta\theta H(x) + i\theta\sigma^l\bar{\theta}(\partial_l A(x)) \\ &\quad - \frac{i}{\sqrt{2}}\theta\theta(\partial_m\psi^{\alpha}(x))\sigma_{\alpha\dot{\alpha}}^m\bar{\theta}^{\dot{\alpha}} + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}(\square A(x)).\end{aligned}\quad (4.1)$$

Under the infinitesimal SUSY transformations (2.3) component fields transform as follows [16]

$$\delta_{\xi}A = \sqrt{2}\xi\psi, \quad (4.2)$$

$$\delta_{\xi}\psi_{\alpha} = i\sqrt{2}\sigma_{\alpha\dot{\alpha}}^m\bar{\xi}^{\dot{\alpha}}(\partial_m A) + \sqrt{2}\xi_{\alpha}H, \quad (4.3)$$

$$\delta_{\xi}H = i\sqrt{2}\bar{\xi}\bar{\sigma}^m(\partial_m\psi). \quad (4.4)$$

The \star -product of two chiral fields reads

$$\begin{aligned}\Phi \star \Phi &= A^2 - \frac{C^2}{2}H^2 + \frac{1}{4}C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma_{\alpha\dot{\alpha}}^m\sigma_{\beta\dot{\beta}}^l(\partial_m A)(\partial_l A) + \frac{1}{64}C^2\bar{C}^2(\square A)^2 \\ &\quad + \theta^{\alpha}\left(2\sqrt{2}\psi_{\alpha}A - \frac{1}{\sqrt{2}}C^{\gamma\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\varepsilon_{\gamma\alpha}(\partial_m\psi^{\rho})\sigma_{\rho\dot{\beta}}^m\sigma_{\beta\dot{\alpha}}^l(\partial_l A)\right) \\ &\quad - \frac{i}{\sqrt{2}}C^2\bar{\theta}_{\dot{\alpha}}\bar{\sigma}^{m\dot{\alpha}\alpha}(\partial_m\psi_{\alpha})H + \theta\theta(2AH - \psi\psi) \\ &\quad + \bar{\theta}\bar{\theta}\left(-\frac{C^2}{4}(H\square A - \frac{1}{2}(\partial_m\psi)\sigma^m\bar{\sigma}^l(\partial_l\psi))\right) \\ &\quad + i\theta\sigma^m\bar{\theta}\left((\partial_m A^2) + \frac{1}{4}C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma_{m\alpha\dot{\alpha}}\sigma_{\beta\dot{\beta}}^l(\square A)(\partial_l A)\right) \\ &\quad + i\sqrt{2}\theta\theta\bar{\theta}_{\dot{\alpha}}\bar{\sigma}^{m\dot{\alpha}\alpha}(\partial_m(\psi_{\alpha}A)) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}(\square A^2),\end{aligned}\quad (4.5)$$

where $C^2 = C^{\alpha\beta}C^{\gamma\delta}\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}$ and $\bar{C}^2 = \bar{C}_{\dot{\alpha}\dot{\beta}}\bar{C}_{\dot{\gamma}\dot{\delta}}\varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\dot{\beta}\dot{\delta}}$. One sees that due to the $\bar{\theta}$ and the $\bar{\theta}\bar{\theta}$ terms (4.5) is not a chiral field. However, in order to write an action invariant under the deformed SUSY transformations (3.11) we need to preserve the notion of chirality. This can be done in different ways. One possibility is to use a different \star -product, the one which preserves chirality [13]. However, chirality-preserving \star -product implies working in Euclidean space where $\bar{\theta} \neq (\theta)^*$. Since we want to work in Minkowski space-time we use the \star -product (3.7) and decompose \star -products of superfields into their irreducible components by using the projectors defined in [16].

The chiral, antichiral and transversal projectors are defined as follows

$$P_1 = \frac{1}{16} \frac{D^2\bar{D}^2}{\square}, \quad (4.6)$$

$$P_2 = \frac{1}{16} \frac{\bar{D}^2D^2}{\square}, \quad (4.7)$$

$$P_T = -\frac{1}{8} \frac{D\bar{D}^2D}{\square}. \quad (4.8)$$

In order to calculate irreducible components of the \star -products of chiral superfields, we first apply the projectors (4.6)–(4.8) to the superfield F (2.2). From the definition of the

supercovariant derivatives

$$D_\alpha = \partial_\alpha + i\sigma_{\alpha\dot{\alpha}}^m \bar{\theta}^{\dot{\alpha}} \partial_m, \quad (4.9)$$

$$\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha \sigma_{\alpha\dot{\alpha}}^m \partial_m, \quad (4.10)$$

follows

$$D^2 = D^\alpha D_\alpha = -\varepsilon^{\alpha\beta} \partial_\alpha \partial_\beta + 2i\varepsilon^{\alpha\beta} \sigma_{\beta\dot{\beta}}^m \bar{\theta}^{\dot{\beta}} \partial_\alpha \partial_m - \bar{\theta}\bar{\theta}\square, \quad (4.11)$$

$$\bar{D}^2 = \bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\partial}_{\dot{\alpha}} \bar{\partial}_{\dot{\beta}} + 2i\theta^\alpha \sigma_{\alpha\dot{\alpha}}^m \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\partial}_{\dot{\beta}} \partial_m - \theta\theta\square. \quad (4.12)$$

Let us start with P_2 and calculate first

$$\begin{aligned} D^2 F &= -4m - 2\bar{\theta}_{\dot{\alpha}} \left(2\bar{\lambda}^{\dot{\alpha}} + i\bar{\sigma}^{m\dot{\alpha}\alpha} (\partial_m \phi_\alpha) \right) + 4i\theta\sigma^l \bar{\theta} (\partial_l m) \\ &\quad - \bar{\theta}\bar{\theta} \left(4d + \square f - 2i(\partial_m v^m) \right) \\ &\quad - \bar{\theta}\bar{\theta}\theta^\alpha \left(2i\sigma_{\alpha\dot{\alpha}}^m (\partial_m \bar{\lambda}^{\dot{\alpha}}) + (\square \phi_\alpha) \right) - \theta\theta\bar{\theta}\bar{\theta} (\square m). \end{aligned} \quad (4.13)$$

Then we have

$$\begin{aligned} \bar{D}^2 D^2 F &= 4 \left(4d + \square f - 2i(\partial_m v^m) \right) + 8\theta^\alpha \left(2i\sigma_{\alpha\dot{\alpha}}^m (\partial_m \bar{\lambda}^{\dot{\alpha}}) + (\square \phi_\alpha) \right) \\ &\quad + 16\theta\theta (\square m) + 4i\theta\sigma^l \bar{\theta} \left(4\partial_l d + \partial_l \square f - 2i(\partial_m \partial_l v^m) \right) \\ &\quad + 4\theta\theta\bar{\theta}_{\dot{\alpha}} \left(2\square \bar{\lambda}^{\dot{\alpha}} + i\bar{\sigma}^{m\dot{\alpha}\alpha} (\partial_m \square \phi_\alpha) \right) \\ &\quad + \theta\theta\bar{\theta}\bar{\theta} \left(4\square d + \square^2 f - 2i\square \partial_m v^m \right). \end{aligned} \quad (4.14)$$

This gives

$$\begin{aligned} P_2 F &= \frac{1}{16} \frac{\bar{D}^2 D^2}{\square} F \\ &= \frac{1}{\square} \left(d - \frac{i}{2} (\partial_m v^m) + \frac{1}{4} \square f \right) + \sqrt{2}\theta^\alpha \left(\frac{i}{\sqrt{2}\square} \sigma_{\alpha\dot{\alpha}}^m (\partial_m \bar{\lambda}^{\dot{\alpha}}) + \frac{1}{2\sqrt{2}} \phi_\alpha \right) \\ &\quad + \theta\theta m + i\theta\sigma^l \bar{\theta} \partial_l \left(\frac{d}{\square} - \frac{i}{2\square} (\partial_m v^m) + \frac{1}{4} f \right) \\ &\quad + \frac{1}{\sqrt{2}} \theta\theta\bar{\theta}_{\dot{\alpha}} \left(\frac{1}{\sqrt{2}} \bar{\lambda}^{\dot{\alpha}} + \frac{i}{2\sqrt{2}} \bar{\sigma}^{m\dot{\alpha}\alpha} (\partial_m \phi_\alpha) \right) + \frac{1}{4} \theta\theta\bar{\theta}\bar{\theta} \left(d - \frac{i}{2} (\partial_m v^m) + \frac{1}{4} \square f \right). \end{aligned} \quad (4.15)$$

The superfield (4.15) is a chiral field with the components

$$\text{scalar: } \mathcal{A} = \frac{1}{\square} \left(d - \frac{i}{2} (\partial_m v^m) + \frac{1}{4} \square f \right), \quad (4.16)$$

$$\text{spinor: } \psi_\alpha = \frac{i}{\sqrt{2}\square} \sigma_{\alpha\dot{\alpha}}^m (\partial_m \bar{\lambda}^{\dot{\alpha}}) + \frac{1}{2\sqrt{2}} \phi_\alpha, \quad (4.17)$$

$$\text{auxiliary field: } \mathcal{H} = m. \quad (4.18)$$

In general, some of these component fields will be nonlocal due to $1/\square$ in the definition of the projector P_2 .

A calculation analogous to the previous one leads to

$$\begin{aligned}
 P_1 F &= \frac{1}{16} \frac{D^2 \bar{D}^2}{\square} F \\
 &= \frac{1}{\square} \left(d + \frac{i}{2} (\partial_m v^m) + \frac{1}{4} \square f \right) + \sqrt{2} \bar{\theta}_{\dot{\alpha}} \left(\frac{i}{\sqrt{2} \square} \bar{\sigma}^{m\dot{\alpha}\alpha} (\partial_m \varphi_\alpha) + \frac{1}{2\sqrt{2}} \bar{\chi}^{\dot{\alpha}} \right) \\
 &\quad + \bar{\theta} \bar{\theta} n - i \theta \sigma^l \bar{\theta} \partial_l \left(\frac{d}{\square} + \frac{i}{2\square} (\partial_m v^m) + \frac{1}{4} f \right) \\
 &\quad - \frac{1}{\sqrt{2}} \bar{\theta} \bar{\theta} \theta^\alpha \left(\frac{1}{\sqrt{2}} \varphi_\alpha - \frac{i}{2\sqrt{2}} \sigma_{\alpha\dot{\alpha}}^m (\partial_m \bar{\chi}^{\dot{\alpha}}) \right) + \frac{1}{4} \theta \theta \bar{\theta} \bar{\theta} \left(d + \frac{i}{2} (\partial_m v^m) + \frac{1}{4} \square f \right),
 \end{aligned} \tag{4.19}$$

which is an antichiral field with the components

$$\text{scalar: } \quad \tilde{\mathcal{A}} = \frac{1}{\square} \left(d + \frac{i}{2} (\partial_m v^m) + \frac{1}{4} \square f \right), \tag{4.20}$$

$$\text{spinor: } \quad \tilde{\psi}^{\dot{\alpha}} = \frac{i}{\sqrt{2} \square} \bar{\sigma}^{m\dot{\alpha}\alpha} (\partial_m \varphi_\alpha) + \frac{1}{2\sqrt{2}} \bar{\chi}^{\dot{\alpha}}, \tag{4.21}$$

$$\text{auxiliary field: } \quad \tilde{\mathcal{H}} = n. \tag{4.22}$$

For the completeness we give the action of the transversal projector P_T on the superfield (2.2). It follows from the identity

$$P_T = I - P_1 - P_2. \tag{4.23}$$

By using (4.15) and (4.19) we obtain

$$\begin{aligned}
 P_T F &= \frac{1}{2} f - \frac{2}{\square} d + \theta^\alpha \left(\frac{1}{2} \phi_\alpha - i \frac{1}{\square} \sigma_{\alpha\dot{\alpha}}^m \partial_m \bar{\lambda}^{\dot{\alpha}} \right) \\
 &\quad + \bar{\theta}_{\dot{\alpha}} \left(\frac{1}{2} \bar{\chi}^{\dot{\alpha}} - i \frac{1}{\square} \bar{\sigma}^{m\dot{\alpha}\alpha} \partial_m \varphi_\alpha \right) + \theta \sigma^m \bar{\theta} \left(v_m - \frac{1}{\square} \partial_m \partial_l v^l \right) \\
 &\quad + \theta \theta \bar{\theta}_{\dot{\alpha}} \left(\frac{1}{2} \bar{\lambda}^{\dot{\alpha}} - \frac{i}{4} \bar{\sigma}^{m\dot{\alpha}\alpha} (\partial_m \phi_\alpha) \right) + \bar{\theta} \bar{\theta} \theta^\alpha \left(\frac{1}{2} \varphi_\alpha - \frac{i}{4} \sigma_{\alpha\dot{\alpha}}^m (\partial_m \bar{\chi}^{\dot{\alpha}}) \right) \\
 &\quad + \frac{1}{4} \theta \theta \bar{\theta} \bar{\theta} \left(2d - \frac{1}{2} \square f \right).
 \end{aligned} \tag{4.24}$$

5. Deformed Wess-Zumino Lagrangian

In the undeformed theory, Wess-Zumino Lagrangian is given by

$$\mathcal{L} = \Phi^+ \cdot \Phi \Big|_{\theta\theta\bar{\theta}\bar{\theta}} + \left(\frac{m}{2} \Phi \cdot \Phi \Big|_{\theta\theta} + \frac{\lambda}{3} \Phi \cdot \Phi \cdot \Phi \Big|_{\theta\theta} + \text{c.c.} \right), \tag{5.1}$$

where m and λ are real constants, Φ is a chiral field and Φ^+ is an antichiral field with $(\Phi^+)^+ = \Phi$. This Lagrangian leads to the SUSY invariant action which describes an interacting theory of two complex scalar fields and one spinor field. To see this explicitly we look at each term separately. This analysis is well known but we repeat it nevertheless to prepare for the analysis of the deformed Wess-Zumino Lagrangian.

The kinetic term is given by the highest component of the product $\Phi^+ \cdot \Phi$:

$$\Phi^+ \cdot \Phi \Big|_{\theta\theta\bar{\theta}\bar{\theta}} = A^* \square A + i(\partial_m \bar{\psi}) \bar{\sigma}^m \psi + H^* H. \quad (5.2)$$

Since $\Phi^+ \cdot \Phi$ is a superfield, its highest component has to transform as a total derivative, (2.14).

Next we look at the mass term. It is given by the $\theta\theta$ component of $\Phi \cdot \Phi$ and the $\bar{\theta}\bar{\theta}$ component of $\Phi^+ \cdot \Phi^+$:

$$\frac{m}{2} \left(\Phi \cdot \Phi \Big|_{\theta\theta} + \Phi^+ \cdot \Phi^+ \Big|_{\bar{\theta}\bar{\theta}} \right) = \frac{m}{2} \left(2AH - \psi\psi + 2A^*H^* - \bar{\psi}\bar{\psi} \right). \quad (5.3)$$

As the pointwise product of two chiral/antichiral fields is a chiral/antichiral field, its $\theta\theta/\bar{\theta}\bar{\theta}$ component transforms as a total derivative (4.4). Note that this is not the case with the general superfield (2.9). Also note that the highest components of $\Phi \cdot \Phi$ and $\Phi^+ \cdot \Phi^+$ transform as total derivatives. However, these terms are total derivatives themselves (4.1) and will not contribute to the equations of motion.

The same arguments apply for the interaction term, since $\Phi \cdot \Phi \cdot \Phi$ is a chiral field again and $\Phi^+ \cdot \Phi^+ \cdot \Phi^+$ is an antichiral field. The interaction term reads

$$\frac{\lambda}{3} \left(\Phi \cdot \Phi \cdot \Phi \Big|_{\theta\theta} + \Phi^+ \cdot \Phi^+ \cdot \Phi^+ \Big|_{\bar{\theta}\bar{\theta}} \right) = \frac{\lambda}{3} \left(HA^2 - A\psi\psi + H^*(A^*)^2 - A^*\bar{\psi}\bar{\psi} \right). \quad (5.4)$$

Thus, we see that chirality plays an important role in the construction of a SUSY invariant action.

We are interested in a deformation of (5.1) which is consistent with the deformed SUSY transformations (3.11) and which in the limit $C^{\alpha\beta} \rightarrow 0$ gives the undeformed Lagrangian (5.1).

We propose the following Lagrangian

$$\mathcal{L} = \Phi^+ \star \Phi \Big|_{\theta\theta\bar{\theta}\bar{\theta}} + \left(\frac{m}{2} P_2(\Phi \star \Phi) \Big|_{\theta\theta} + \frac{\lambda}{3} P_2(\Phi \star P_2(\Phi \star \Phi)) \Big|_{\theta\theta} + \text{c.c.} \right), \quad (5.5)$$

where m and λ are real constants. Let us analyse (5.5) term by term again.

Kinetic term in (5.5) is a straightforward deformation of the usual kinetic term obtained by inserting the \star -product instead the usual pointwise multiplication. Due to the deformed coproduct (3.3), $\Phi^+ \star \Phi$ is a superfield and its highest component transforms as a total derivative. The explicit calculation gives

$$\begin{aligned} \Phi^+ \star \Phi \Big|_{\theta\theta\bar{\theta}\bar{\theta}} &= A^* \square A + i(\partial_m \bar{\psi}) \bar{\sigma}^m \psi + H^* H, \quad (5.6) \\ \delta_\xi^* \left(\Phi^+ \star \Phi \Big|_{\theta\theta\bar{\theta}\bar{\theta}} \right) &= \partial_m \left(\frac{1}{2\sqrt{2}} (A^* (\partial_l \psi^\alpha) - (\partial_l A^*) \psi^\alpha) (\sigma^l \bar{\sigma}^m)_\alpha{}^\beta + \frac{i}{\sqrt{2}} H \bar{\psi}_\alpha \bar{\sigma}^{m\dot{\alpha}\beta} \right) \xi_\beta \\ &\quad + \bar{\xi}_{\dot{\alpha}} \partial_m \left(\frac{1}{2\sqrt{2}} (\bar{\sigma}^m \sigma^l)^{\dot{\alpha}\beta} (\bar{\psi}^{\dot{\beta}} (\partial_l A) - (\partial_l \bar{\psi}^{\dot{\beta}}) A) + \frac{i}{\sqrt{2}} \bar{\sigma}^{m\dot{\alpha}\alpha} H^* \psi_\alpha \right). \quad (5.7) \end{aligned}$$

To obtain (5.6), the partial integration was used. We see from (5.6) that the deformation is absent, the kinetic term remains undeformed.¹

¹In the case of the Moyal-Weyl \star -product we have $\int d^4x f \star_{\text{mw}} g = \int d^4x g \star_{\text{mw}} f = \int d^4x f \cdot g$. Therefore, the free actions for scalar and spinor fields remain undeformed automatically.

Since $\Phi \star \Phi$ is not a chiral field we have to project its chiral part. This projection is given by

$$\begin{aligned}
 P_2(\Phi \star \Phi) = & A^2 - \frac{C^2}{8}H^2 + \frac{1}{256}C^2\bar{C}^2(\square A)^2 \\
 & + \frac{1}{16}C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^m_{\alpha\dot{\alpha}}\sigma^l_{\beta\dot{\beta}}\left((\partial_m A)(\partial_l A) + \frac{2}{\square}\partial_m((\square A)(\partial_l A))\right) \\
 & + \sqrt{2}\theta^\alpha\left(2\psi_\alpha A - \frac{1}{4}C^{\gamma\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\varepsilon_{\gamma\alpha}(\partial_m\psi^\rho)\sigma^m_{\rho\dot{\beta}}\sigma^l_{\beta\dot{\alpha}}(\partial_l A)\right) + \theta\theta(2AH - \psi\psi) \\
 & + i\theta\sigma^k\bar{\theta}\partial_k\left[A^2 - \frac{C^2}{8}H^2 + \frac{1}{256}C^2\bar{C}^2(\square A)^2\right. \\
 & \quad \left.+ \frac{1}{16}C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^m_{\alpha\dot{\alpha}}\sigma^l_{\beta\dot{\beta}}\left((\partial_m A)(\partial_l A) + \frac{2}{\square}\partial_m((\square A)(\partial_l A))\right)\right] \\
 & + i\sqrt{2}\theta\theta\bar{\theta}_\alpha\bar{\sigma}^{k\dot{\alpha}\alpha}\partial_k\left(\psi_\alpha A - \frac{1}{8}C^{\gamma\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\varepsilon_{\gamma\alpha}(\partial_m\psi^\rho)\sigma^m_{\rho\dot{\beta}}\sigma^l_{\beta\dot{\alpha}}(\partial_l A)\right) \\
 & + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\square\left[A^2 - \frac{C^2}{8}H^2 + \frac{1}{256}C^2\bar{C}^2(\square A)^2\right. \\
 & \quad \left.+ \frac{1}{16}C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^m_{\alpha\dot{\alpha}}\sigma^l_{\beta\dot{\beta}}\left((\partial_m A)(\partial_l A) + \frac{2}{\square}\partial_m((\square A)(\partial_l A))\right)\right]. \quad (5.8)
 \end{aligned}$$

For the action we take the $\theta\theta$ component of (5.8),

$$P_2(\Phi \star \Phi)\Big|_{\theta\theta} = 2AH - \psi\psi. \quad (5.9)$$

Its transformation law is given by

$$\delta_\xi^*(P_2(\Phi \star \Phi)\Big|_{\theta\theta}) = 2i\sqrt{2}\bar{\xi}\bar{\sigma}^m\partial_m(A\psi). \quad (5.10)$$

In a similar way we add the $\bar{\theta}\bar{\theta}$ component of $P_1(\Phi^+ \star \Phi^+)$. This component is given by

$$P_1(\Phi^+ \star \Phi^+)\Big|_{\bar{\theta}\bar{\theta}} = 2A^*H - \bar{\psi}\bar{\psi}, \quad (5.11)$$

which is just the complex conjugate of (5.9) due to the hermiticity of the \star -product (3.7). Again, no deformation is present: the free action remains undeformed. That leads to the propagators which are the same as in the undeformed theory.

Finally we come to the interaction term. There are few possibilities to project the chiral part of $\Phi \star \Phi \star \Phi$. We take the following projection²

$$\Phi \star \Phi \star \Phi \rightarrow P_2(\Phi \star (P_2(\Phi \star \Phi))). \quad (5.12)$$

²Naively, one would take $P_2(\Phi \star \Phi \star \Phi)\Big|_{\theta\theta}$. Despite the fact that $P_2(\Phi \star \Phi \star \Phi)$ is a chiral field, its $\theta\theta$ component does not transform as a total derivative and would not lead to a SUSY invariant action. This strange situation arises because of the $1/\square$ term in the projector P_2 .

As the complete result is very long we write here only the $\theta\theta$ component

$$\begin{aligned}
 P_2\left(\Phi \star (P_2(\Phi \star \Phi))\right)\Big|_{\theta\theta} &= 3(A^2 H - (\psi\psi)A) - \frac{C^2}{8}H^3 + \frac{1}{256}C^2\bar{C}^2 H(\square A)^2 \\
 &+ \frac{1}{16}C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma_{\alpha\dot{\alpha}}^m\sigma_{\beta\dot{\beta}}^l H\left((\partial_m A)(\partial_l A) + \frac{2}{\square}\partial_m((\square A)(\partial_l A))\right) \\
 &+ \frac{1}{4}C^{\gamma\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}H\sigma_{\beta\dot{\alpha}}^l\psi_\gamma(\partial_m\psi^\rho)\sigma_{\rho\dot{\beta}}^m(\partial_l A) \\
 &+ \frac{1}{2}\bar{C}^{\dot{\alpha}\dot{\beta}}(\bar{\sigma}^{lm})^{\dot{\beta}}_{\dot{\gamma}}\varepsilon^{\dot{\gamma}\dot{\alpha}}(\partial_m A)\partial_l\left[A^2 - \frac{C^2}{8}H^2\right. \\
 &\left.+ \frac{1}{16}C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma_{\alpha\dot{\alpha}}^s\sigma_{\beta\dot{\beta}}^p H\left((\partial_s A)(\partial_p A) + \frac{2}{\square}\partial_s((\square A)(\partial_p A))\right)\right].
 \end{aligned} \tag{5.13}$$

In the limit $C^{\alpha\beta} \rightarrow 0$ (5.13) reduces to the usual interaction term (5.4). The deformation is present through the terms that are of first, second and higher orders in $C^{\alpha\beta}$ and $\bar{C}^{\dot{\alpha}\dot{\beta}}$. Note that under the integral the last term reduces to a total derivative and therefore will not contribute to the equations of motion. Also note that if we calculate $P_2\left((P_2(\Phi \star \Phi)) \star \Phi\right)$ instead of (5.12) the only difference will be in the sign of the above-mentioned last term. We therefore conclude that we can take any combination of these two terms, as long as the limit $C^{\alpha\beta} \rightarrow 0$ reproduces the undeformed interaction term. For simplicity we take only (5.13).

The transformation law of (5.13) is given by

$$\begin{aligned}
 \delta_\xi^*\left(P_2\left(\Phi \star (P_2(\Phi \star \Phi))\right)\Big|_{\theta\theta}\right) &= \\
 &i\sqrt{2}\bar{\xi}_{\dot{\alpha}}\bar{\sigma}^{l\dot{\alpha}\alpha}\partial_l\left(\frac{1}{8}C^{\gamma\beta}\bar{C}^{\dot{\gamma}\dot{\beta}}\sigma_{\gamma\dot{\gamma}}^m\sigma_{\beta\dot{\beta}}^n\psi_\alpha\frac{1}{\square}\partial_m(\partial_n A\square A) + \text{local terms}\right).
 \end{aligned} \tag{5.14}$$

The SUSY transformation is a total derivative and reduces to a surface term under the integral, leading to a SUSY invariant interaction term. However, one should be careful as (5.14) contains a non-local term. Under the integral it is proportional to

$$\int d^4x \bar{\sigma}^{l\dot{\alpha}\alpha}\partial_l\left(\psi_\alpha\frac{1}{\square}\partial_m(\partial_n A\square A)\right) = \oint d\Sigma_l \bar{\sigma}^{l\dot{\alpha}\alpha}\left(\psi_\alpha\frac{1}{\square}\partial_m(\partial_n A\square A)\right).$$

If the boundary surface Σ_l is at infinity and fields fall off fast enough this integral vanishes.

To rewrite (5.13) in a more compact way we introduce the following notation

$$C_{\alpha\beta} = K_{ab}(\sigma^{ab}\varepsilon)_{\alpha\beta}, \tag{5.15}$$

$$\bar{C}^{\dot{\alpha}\dot{\beta}} = K_{ab}^*(\varepsilon\bar{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}}, \tag{5.16}$$

where $K_{ab} = -K_{ba}$ is an antisymmetric complex constant matrix. Then we have

$$C^2 = 2K_{ab}K^{ab}, \quad \bar{C}^2 = 2K_{ab}^*K^{*ab}, \quad K^{ab}K_{ab}^* = 0, \tag{5.17}$$

$$K_{cd}^*K_{ab}(\sigma^n\bar{\sigma}^{cd}\bar{\sigma}^m\sigma^{ab})_\alpha^\beta = -4\delta_\alpha^\beta K^{ma}K_a^{*n} + 8K^{ma}K^{*nb}(\sigma_{ba})_\alpha^\beta, \tag{5.18}$$

$$C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma_{\alpha\dot{\alpha}}^m\sigma_{\beta\dot{\beta}}^l = 8K^{am}K_a^{*l}. \tag{5.19}$$

By using the previous expressions the term (5.13) can be rewritten in the form

$$\begin{aligned}
P_2\left(\Phi \star (P_2(\Phi \star \Phi))\right)\Big|_{\theta\theta} &= 3(A^2 H - (\psi\psi)A) - \frac{1}{4}K^{ab}K_{ab}H^3 \\
&+ \frac{1}{64}K^{ab}K_{ab}K^{*cd}K_{cd}^*H(\square A)^2 \\
&+ \frac{1}{2}K^m{}_l K^{*nl}H\left((\partial_m A)(\partial_n A) + \frac{2}{\square}\partial_m((\square A)(\partial_n A))\right) \\
&- \left(K^m{}_l K^{*nl}\psi(\partial_n\psi) - 2K^m{}_a K^{*n}{}_c(\partial_n\psi)\sigma^{ca}\psi\right)(\partial_m A).
\end{aligned} \tag{5.20}$$

Finally, the deformed SUSY invariant Lagrangian is given by

$$\begin{aligned}
\mathcal{L} &= \Phi^+ \star \Phi\Big|_{\theta\theta\bar{\theta}\bar{\theta}} + \left(\frac{m}{2}P_2(\Phi \star \Phi)\Big|_{\theta\theta} + \frac{\lambda}{3}P_2\left(\Phi \star P_2(\Phi \star \Phi)\right)\Big|_{\theta\theta} + \text{c.c.}\right) \\
&= A^*\square A + i(\partial_m\bar{\psi})\bar{\sigma}^m\psi + H^*H \\
&+ \frac{m}{2}\left(2AH - \psi\psi + 2A^*H^* - \bar{\psi}\bar{\psi}\right) \\
&+ \lambda\left(HA^2 - A\psi\psi + H^*(A^*)^2 - A^*\bar{\psi}\bar{\psi}\right) \\
&- \frac{\lambda}{3}\left(K^m{}_a K^{*na}\psi(\partial_n\psi) - 2K^m{}_a K^{*n}{}_b(\partial_n\psi)\sigma^{ba}\psi\right)(\partial_m A) \\
&- \frac{\lambda}{3}\left(K^m{}_a K^{*na}\bar{\psi}(\partial_n\bar{\psi}) - 2K^{*m}{}_a K^n{}_b\bar{\psi}\bar{\sigma}^{ab}(\partial_n\bar{\psi})\right)(\partial_m A^*) \\
&- \frac{\lambda}{12}K^{mn}K_{mn}H^3 - \frac{\lambda}{12}K^{*mn}K_{mn}^*(H^*)^3 \\
&+ \frac{\lambda}{6}K^m{}_l K^{*nl}\left(H(\partial_m A)(\partial_n A) + H^*(\partial_m A^*)(\partial_n A^*)\right) \\
&+ \frac{\lambda}{3}K^m{}_l K^{*nl}\left[H\frac{1}{\square}\partial_m((\partial_n A)\square A) + H^*\frac{1}{\square}\partial_m((\partial_n A^*)\square A^*)\right] \\
&+ \frac{\lambda}{192}K^{ab}K_{ab}K^{*cd}K_{cd}^*\left(H(\square A)^2 + H^*(\square A^*)^2\right),
\end{aligned} \tag{5.21}$$

where the partial integration was used to rewrite some of the terms in (5.21) in a more compact way.

6. Equations of motion

By varying the action which follows from the Lagrangian (5.21) with respect to the fields H and H^* we obtain the equations of motion

$$\begin{aligned}
H^* + mA + \lambda A^2 - \frac{\lambda}{4}K^{ab}K_{ab}H^2 + \frac{\lambda}{6}K^m{}_l K^{*nl}(\partial_m A)(\partial_n A) \\
+ \frac{\lambda}{3}K^m{}_l K^{*nl}\frac{1}{\square}\partial_m((\partial_n A)\square A) + \frac{\lambda}{192}K^{ab}K_{ab}K^{*cd}K_{cd}^*(\square A)^2 = 0,
\end{aligned} \tag{6.1}$$

$$\begin{aligned}
H + mA^* + \lambda(A^*)^2 - \frac{\lambda}{4}K^{*cd}K_{cd}^*(H^*)^2 + \frac{\lambda}{6}K^m{}_l K^{*nl}(\partial_m A^*)(\partial_n A^*) \\
+ \frac{\lambda}{3}K^m{}_l K^{*nl}\frac{1}{\square}\partial_m((\partial_n A^*)\square A^*) + \frac{\lambda}{192}K^{ab}K_{ab}K^{*cd}K_{cd}^*(\square A^*)^2 = 0.
\end{aligned} \tag{6.2}$$

Unlike the undeformed theory, equations (6.1) and (6.2) are nonlinear in H and H^* . Nevertheless, they can be solved perturbatively. The solutions are given by

$$\begin{aligned}
 H^* = & -mA - \lambda A^2 + \frac{\lambda}{4} K^{ab} K_{ab} (mA^* + \lambda(A^*)^2)^2 \\
 & - \frac{\lambda}{6} K^m{}_l K^{*nl} (\partial_m A) (\partial_n A) - \frac{\lambda}{3} K^m{}_l K^{*nl} \frac{1}{\square} \partial_m ((\partial_n A) \square A) \\
 & - \frac{\lambda}{192} K^{ab} K_{ab} K^{*cd} K_{cd}^* (\square A)^2 \\
 & + \frac{\lambda}{2} K^{ab} K_{ab} (mA^* + \lambda(A^*)^2) \left[\frac{\lambda}{6} K^m{}_l K^{*nl} (\partial_m A^*) (\partial_n A^*) \right. \\
 & \left. + \frac{\lambda}{3} K^m{}_l K^{*nl} \frac{1}{\square} \partial_m ((\partial_n A^*) \square A^*) + \frac{\lambda}{4} K^{*cd} K_{cd}^* (mA + \lambda A^2)^2 \right] + \mathcal{O}(K^6),
 \end{aligned} \tag{6.3}$$

$$\begin{aligned}
 H = & -mA^* - \lambda(A^*)^2 + \frac{\lambda}{4} K^{*cd} K_{cd}^* (mA + \lambda A^2)^2 - \frac{\lambda}{6} K^m{}_l K^{*nl} (\partial_m A^*) (\partial_n A^*) \\
 & - \frac{\lambda}{3} K^m{}_l K^{*nl} \frac{1}{\square} \partial_m ((\partial_n A^*) \square A^*) - \frac{\lambda}{192} K^{ab} K_{ab} K^{*cd} K_{cd}^* (\square A^*)^2 \\
 & + \frac{\lambda}{2} K^{*cd} K_{cd}^* (mA + \lambda A^2) \left[\frac{\lambda}{6} K^m{}_l K^{*nl} (\partial_m A) (\partial_n A) \right. \\
 & \left. + \frac{\lambda}{3} K^m{}_l K^{*nl} \frac{1}{\square} \partial_m ((\partial_n A) \square A) + \frac{\lambda}{4} K^{ab} K_{ab} (mA^* + \lambda(A^*)^2)^2 \right] + \mathcal{O}(K^6).
 \end{aligned} \tag{6.4}$$

These solutions can be used to eliminate the auxiliary fields H and H^* from the Lagrangian (5.21). This gives

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{O}(K^6), \tag{6.5}$$

with

$$\begin{aligned}
 \mathcal{L}_0 = & A^* \square A + i(\partial_m \bar{\psi}) \bar{\sigma}^m \psi - \lambda A^* \bar{\psi} \bar{\psi} - \lambda A \psi \psi - \frac{m}{2} (\psi \psi + \bar{\psi} \bar{\psi}) \\
 & - m^2 A^* A - m \lambda A (A^*)^2 - m \lambda A^* A^2 - \lambda^2 A^2 (A^*)^2,
 \end{aligned} \tag{6.6}$$

$$\begin{aligned}
 \mathcal{L}_2 = & \frac{\lambda}{3} K^m{}_l K^{*nl} \left(m(\partial_m A) + 2\lambda A (\partial_m A) \right) \frac{1}{\square} ((\partial_n A^*) \square A^*) \\
 & + \frac{\lambda}{3} K^m{}_l K^{*nl} \left(m(\partial_m A^*) + 2\lambda A^* (\partial_m A^*) \right) \frac{1}{\square} ((\partial_n A) \square A) \\
 & + \frac{\lambda}{12} K^{ab} K_{ab} (mA^* + \lambda(A^*)^2)^3 + \frac{\lambda}{12} K^{*cd} K_{cd}^* (mA + \lambda A^2)^3 \\
 & - \frac{\lambda}{6} K^m{}_l K^{*nl} \left((mA + \lambda A^2) (\partial_m A^*) (\partial_n A^*) + (mA^* + \lambda(A^*)^2) (\partial_m A) (\partial_n A) \right) \\
 & - \frac{\lambda}{3} \left(K^m{}_l K^{*nl} \psi (\partial_n \psi) - 2K^m{}_a K^{*n}{}_b (\partial_n \psi) \sigma^{ba} \psi \right) (\partial_m A) \\
 & - \frac{\lambda}{3} \left(K^m{}_l K^{*nl} \bar{\psi} (\partial_n \bar{\psi}) - 2K^m{}_a K^{*n}{}_b \bar{\psi} \bar{\sigma}^{ab} (\partial_n \bar{\psi}) \right) (\partial_m A^*),
 \end{aligned} \tag{6.7}$$

$$\begin{aligned}
 \mathcal{L}_4 = & \frac{\lambda^2}{24} K^m{}_l K^{*nl} K^{ab} K_{ab} (mA^* + \lambda(A^*)^2)^2 (\partial_m A^*) (\partial_n A^*) \\
 & + \frac{\lambda^2}{24} K^m{}_l K^{*nl} K^{*cd} K_{cd}^* (mA + \lambda A^2)^2 (\partial_m A) (\partial_n A)
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{\lambda}{192} K^{ab} K_{ab} K^{*mn} K_{mn}^* (mA + \lambda A^2) (\square A^*)^2 \\
 & -\frac{\lambda}{192} K^{ab} K_{ab} K^{*mn} K_{mn}^* (mA^* + \lambda (A^*)^2) (\square A)^2 \\
 & -\frac{\lambda}{16} K^{ab} K_{ab} K^{*cd} K_{cd}^* (mA + \lambda A^2)^2 (mA^* + \lambda (A^*)^2)^2 \\
 & -\frac{\lambda^2}{18} K^m_l K^{*nl} K^{pb} K^{*q}_b \left((\partial_m A^*) (\partial_n A^*) \right) \frac{1}{\square} \partial_p \left((\partial_q A) \square A \right) \\
 & -\frac{\lambda^2}{18} K^m_l K^{*nl} K^{pb} K^{*q}_b \left((\partial_m A) (\partial_n A) \right) \frac{1}{\square} \partial_p \left((\partial_q A^*) \square A^* \right) \\
 & -\frac{\lambda^2}{6} K^m_l K^{*nl} K^{ab} K_{ab} (mA^* + \lambda (A^*)^2) \left(m (\partial_m A^*) + 2\lambda A^* (\partial_m A^*) \right) \frac{1}{\square} \left((\partial_n A^*) \square A^* \right) \\
 & -\frac{\lambda^2}{6} K^m_l K^{*nl} K^{*cd} K_{cd}^* (mA + \lambda A^2) \left(m (\partial_m A) + 2\lambda A (\partial_m A) \right) \frac{1}{\square} \left((\partial_n A) \square A \right) \\
 & -\frac{\lambda^2}{9} K^m_l K^{*nl} K^p_b K^{*qb} \frac{1}{\square} \partial_m \left((\partial_n A) \square A \right) \frac{1}{\square} \partial_p \left((\partial_q A^*) \square A^* \right) \\
 & -\frac{\lambda^2}{36} K^m_l K^{*nl} K^p_b K^{*qb} (\partial_m A) (\partial_n A) (\partial_p A^*) (\partial_q A^*). \tag{6.8}
 \end{aligned}$$

7. Deformed Poincaré invariance

Before commenting on the Lagrangian (6.5) we shall analyze the consequences of the twist (3.1) on Poincaré symmetry. As in the case of the θ -deformed space, the sub(Hopf)algebra of translations remains undeformed [22]. Therefore we concentrate on the Lorentz transformations and first review some well known facts and formulas.

Under the infinitesimal Lorentz transformations the coordinates of the superspace transform as follows

$$\delta_\omega x^m = \omega^n_m x^n, \tag{7.1}$$

$$\delta_\omega \theta_\alpha = \omega^{mn} (\sigma_{mn})_\alpha^\beta \theta_\beta, \tag{7.2}$$

$$\delta_\omega \bar{\theta}^{\dot{\alpha}} = \omega^{mn} (\bar{\sigma}_{mn})_{\dot{\alpha}}^{\dot{\beta}} \bar{\theta}^{\dot{\beta}}, \tag{7.3}$$

where $\omega^{mn} = -\omega^{nm}$ are constant antisymmetric parameters.

The superfield F (2.2) is a scalar under the Lorentz transformations

$$F'(x', \theta', \bar{\theta}') = F(x, \theta, \bar{\theta}), \tag{7.4}$$

or

$$\begin{aligned}
 \delta_\omega F &= F'(x, \theta, \bar{\theta}) - F(x, \theta, \bar{\theta}) \\
 &= \frac{1}{2} \omega^{mn} L_{mn} F(x, \theta, \bar{\theta}) \\
 &= \frac{1}{2} \omega^{mn} \left(x_m \partial_n - x_n \partial_m - (\sigma_{mn} \varepsilon)_{\alpha\beta} (\theta^\alpha \partial^\beta + \theta^\beta \partial^\alpha) \right. \\
 &\quad \left. - (\varepsilon \bar{\sigma}_{mn})_{\dot{\alpha}\dot{\beta}} (\bar{\theta}^{\dot{\alpha}} \bar{\partial}^{\dot{\beta}} + \bar{\theta}^{\dot{\beta}} \bar{\partial}^{\dot{\alpha}}) \right) F(x, \theta, \bar{\theta}). \tag{7.5}
 \end{aligned}$$

To calculate the last line in (7.5) we used (7.1), (7.2) and (7.3). Note that we use the same notation for transformations of coordinates and for variation of fields. The meaning should be clear from the context. Using the generators L_{mn} we can rewrite (7.1), (7.2) and (7.3) in the following way

$$\delta_\omega x^m = \omega^m_n x^n = -\frac{1}{2}\omega^{rs} L_{rs} x^m, \quad (7.6)$$

$$\delta_\omega \theta_\alpha = \omega^{mn} (\sigma_{mn})_\alpha^\beta \theta_\beta = -\frac{1}{2}\omega^{mn} L_{mn} \theta_\alpha, \quad (7.7)$$

$$\delta_\omega \bar{\theta}^{\dot{\alpha}} = \omega^{mn} (\bar{\sigma}_{mn})_{\dot{\beta}}^{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} = -\frac{1}{2}\omega^{mn} L_{mn} \bar{\theta}^{\dot{\alpha}}. \quad (7.8)$$

Also,

$$\delta_\omega \theta^\alpha = -\omega^{mn} (\sigma_{mn})_\beta^\alpha \theta^\beta = -\frac{1}{2}\omega^{mn} L_{mn} \theta^\alpha. \quad (7.9)$$

The Hopf algebra of the undeformed infinitesimal Lorentz transformations is given by

$$\begin{aligned} [\delta_\omega, \delta_{\omega'}] &= \delta_{[\omega, \omega']}, \\ \Delta(\delta_\omega) &= \delta_\omega \otimes 1 + 1 \otimes \delta_\omega, \\ \varepsilon(\delta_\omega) &= 0, \quad S(\delta_\omega) = -\delta_\omega. \end{aligned} \quad (7.10)$$

In terms of the generator L_{mn} the coproduct reads

$$\Delta(L_{mn}) = L_{mn} \otimes 1 + 1 \otimes L_{mn}. \quad (7.11)$$

The twist \mathcal{F} (3.1), when applied to (7.10), gives the Hopf algebra of the deformed Lorentz transformations

$$\begin{aligned} [\delta_\omega, \delta_{\omega'}] &= \delta_{[\omega, \omega']}, \\ \Delta_{\mathcal{F}}(\delta_\omega) &= \mathcal{F}(\delta_\omega \otimes 1 + 1 \otimes \delta_\omega) \mathcal{F}^{-1} \\ &= \delta_\omega \otimes 1 + 1 \otimes \delta_\omega \\ &\quad - \frac{1}{2} C^{\alpha\beta} \omega^{mn} (\partial_\alpha \otimes (\sigma_{mn})_{\beta\gamma} \partial^\gamma + (\sigma_{mn})_{\alpha\gamma} \partial^\gamma \otimes \partial_\beta) \\ &\quad - \frac{1}{2} \bar{C}_{\dot{\alpha}\dot{\beta}} \omega^{mn} (\bar{\partial}^{\dot{\alpha}} \otimes (\varepsilon \bar{\sigma}_{mn})_{\dot{\rho}\dot{\sigma}} \varepsilon^{\dot{\sigma}\dot{\beta}} \bar{\partial}^{\dot{\rho}} + (\varepsilon \bar{\sigma}_{mn})_{\dot{\rho}\dot{\sigma}} \varepsilon^{\dot{\sigma}\dot{\alpha}} \bar{\partial}^{\dot{\rho}} \otimes \bar{\partial}^{\dot{\beta}}), \\ \varepsilon(\delta_\omega) &= 0, \quad S(\delta_\omega) = -\delta_\omega. \end{aligned} \quad (7.12)$$

The result for the deformed coproduct is the result to all orders, as all higher order terms cancel since transformations (7.5) are linear in coordinates. The algebra is unchanged, but the comultiplication, leading to the deformed Leibniz rule, changes. Form (7.12) one can see that the comultiplication for the deformed Lorentz transformations does not close in the algebra of Lorentz transformations, but in the bigger algebra with derivatives included. Therefore, we cannot speak about the deformed Lorentz symmetry but instead we have to work with the deformed Poincaré symmetry.

Now we give two examples for the application of the deformed Leibniz rule.

- The \star -product of two Grassmanian coordinates should transform as in the undeformed case

$$\begin{aligned}
 \delta_\omega(\theta^\alpha \star \theta^\beta) &= -\frac{1}{2}\omega^{mn}L_{mn}(\theta^\alpha \star \theta^\beta) \\
 &= \frac{1}{2}\omega^{mn}(\sigma_{mn}\varepsilon)_{\gamma\delta}(\theta^\gamma\partial^\delta + \theta^\delta\partial^\gamma)\left(\theta^\alpha\theta^\beta + \frac{1}{2}C^{\alpha\beta}\right) \\
 &= -\omega^{mn}\left((\sigma_{mn})_\gamma^\alpha\theta^\gamma\theta_\beta + (\sigma_{mn})_\gamma^\beta\theta_\alpha\theta^\gamma\right). \tag{7.13}
 \end{aligned}$$

In the second line the \star -product is expanded and the definition of L_{mn} given in (7.5) is used. Using the deformed coproduct on the other hand gives

$$\begin{aligned}
 \delta_\omega(\theta^\alpha \star \theta^\beta) &= (\delta_\omega\theta^\alpha) \star \theta^\beta + \theta^\alpha \star (\delta_\omega\theta^\beta) \\
 &\quad -\frac{1}{2}C^{\rho\sigma}\omega^{mn}\left((\partial_\rho\theta^\alpha) \star (\sigma_{mn}\varepsilon)_{\sigma\gamma}(\partial^\gamma\theta^\beta)\right. \\
 &\quad \left.+(\sigma_{mn}\varepsilon)_{\rho\gamma}(\partial^\gamma\theta^\alpha) \star (\partial_\sigma\theta^\beta)\right) \\
 &= -\omega^{mn}\left((\sigma_{mn})_\gamma^\alpha\theta^\gamma\theta_\beta + (\sigma_{mn})_\gamma^\beta\theta_\alpha\theta^\gamma\right). \tag{7.14}
 \end{aligned}$$

Comparing the results (7.13) and (7.14) we see that due to the deformed coproduct $\theta^\alpha \star \theta^\beta$ transforms as in the undeformed case. This type of calculation can also be done for \star -products of $\bar{\theta}$ coordinates with the same conclusions.

- When the \star -product of two chiral fields Φ_1 and Φ_2 is expanded, the term $C^{\alpha\beta}\psi_{1\alpha}\psi_{2\beta}$ appears. This term has to transform as a scalar field under the deformed Poncaré transformations, since it comes from $\Phi_1\star\Phi_2$ which is a scalar field (using the deformed Leibniz rule of course).

Naively we have

$$\begin{aligned}
 \delta_\omega(C^{\alpha\beta}\psi_{1\alpha}\psi_{2\beta}) &= C^{\alpha\beta}\left((\delta_\omega\psi_{1\alpha})\psi_{2\beta} + \psi_{1\alpha}(\delta_\omega\psi_{2\beta})\right) \\
 &= C^{\alpha\beta}\omega^{mn}\left((\sigma_{mn})_\alpha^\gamma\psi_{1\gamma}\psi_{2\beta} + (\sigma_{mn})_\beta^\gamma\psi_{1\alpha}\psi_{2\gamma}\right. \\
 &\quad \left.+\frac{1}{2}(x_m\partial_n - x_n\partial_m)(\psi_{1\alpha}\psi_{2\beta})\right) \\
 &\neq \frac{1}{2}\omega^{mn}L_{mn}(C^{\alpha\beta}\psi_{1\alpha}\psi_{2\beta}), \tag{7.15}
 \end{aligned}$$

with L_{mn} defined in (7.5). The equality sign in the last line can be achieved by transforming the fields $\psi_{1\alpha}$ and $\psi_{2\beta}$ not as spinor fields (as it was done in (7.15)) but as scalar fields. The reason for this is that indices α and β are contracted with indices on $C^{\alpha\beta}$. Namely, the twist \mathcal{F} (3.1) is a globally defined object [23]. Therefore, under the transformations (7.2) and (7.3) the derivatives ∂ and $\bar{\partial}$ appearing in \mathcal{F} transform in the following way

$$\delta_\omega\partial_\alpha = \delta_\omega\bar{\partial}_\alpha = 0. \tag{7.16}$$

Also, $C^{\alpha\beta}$ and $\bar{C}^{\dot{\alpha}\dot{\beta}}$ (being complex constants) do not transform. Therefore, all indices contracted with $C^{\alpha\beta}$ and $\bar{C}^{\dot{\alpha}\dot{\beta}}$ should be understood as scalar (non-transforming) indices.

To convince ourselves that this is the right way of thinking let us rewrite $C^{\alpha\beta}\psi_{1\alpha}\psi_{2\beta}$ by using the \star -product and then use the deformed Leibniz rule to transform it

$$\begin{aligned} C^{\alpha\beta}\psi_{1\alpha}\psi_{2\beta} &= -2\theta^\alpha\psi_{1\alpha}\star\theta^\beta\psi_{2\beta} - \theta\theta\psi_1^\alpha\psi_{2\alpha} \\ &= -2\theta^\alpha\psi_{1\alpha}\star\theta^\beta\psi_{2\beta} - (\theta^\alpha\star\theta_\alpha)\psi_1^\beta\psi_{2\beta} \\ \delta_\omega(C^{\alpha\beta}\psi_{1\alpha}\psi_{2\beta}) &= -2\delta_\omega(\theta^\alpha\psi_{1\alpha}\star\theta^\beta\psi_{2\beta}) - \delta_\omega((\theta^\alpha\star\theta_\alpha)\psi_1^\beta\psi_{2\beta}). \end{aligned} \quad (7.17)$$

Note that $\psi_1^\beta\star\psi_{2\beta} = \psi_1^\beta\psi_{2\beta}$. Also note that δ_ω in this example is the variation of a field as in (7.5). Therefore

$$\begin{aligned} \delta_\omega(\theta^\alpha\psi_{1\alpha}) &= \theta^\alpha\delta_\omega(\psi_{1\alpha}) \\ &= \frac{1}{2}\omega^{mn}L_{mn}(\theta^\alpha\psi_{1\alpha}). \end{aligned}$$

Let us calculate the transformation of the first term in (7.17)

$$\begin{aligned} \delta_\omega(\theta^\alpha\psi_{1\alpha}\star\theta^\beta\psi_{2\beta}) &= (\delta_\omega(\theta^\alpha\psi_{1\alpha}))\star(\theta^\beta\psi_{2\beta}) + (\theta^\alpha\psi_{1\alpha})\star(\delta_\omega(\theta^\beta\psi_{2\beta})) \\ &\quad - \frac{1}{2}C^{\rho\sigma}\omega^{mn}\left((\partial_\rho(\theta^\alpha\psi_{1\alpha}))\star(\sigma_{mn}\varepsilon)_{\sigma\gamma}(\partial^\gamma(\theta^\beta\psi_{2\beta}))\right. \\ &\quad \left.+ (\sigma_{mn}\varepsilon)_{\rho\gamma}(\partial^\gamma(\theta^\alpha\psi_{1\alpha}))\star(\partial_\sigma(\theta^\beta\psi_{2\beta}))\right) \\ &= \frac{1}{2}\omega^{mn}L_{mn}\left(\theta^\alpha\psi_{1\alpha}\star\theta^\beta\psi_{2\beta}\right). \end{aligned} \quad (7.18)$$

We conclude that $\theta^\alpha\psi_{1\alpha}\star\theta^\beta\psi_{2\beta}$ is a scalar field. Calculation similar to this shows that $(\theta^\alpha\star\theta_\alpha)\psi_1^\beta\psi_{2\beta}$ is also a scalar field. Thus, we have demonstrated that $C^{\alpha\beta}\psi_{1\alpha}\psi_{2\beta}$ really transforms as a scalar field.

8. Conclusions and outlook

The Lagrangian (6.5) is the final result of this paper. By construction this Lagrangian is covariant under the deformed SUSY transformations (3.11) and leads to a deformed SUSY invariant action. Note that it is the deformed Leibniz rule which enables this construction. No new fields appear in the action, the deformation is present only through some new interaction terms. The deformation parameter plays the role of a coupling constant and in the limit $C \rightarrow 0$ the undeformed theory is obtained. If this leads to some new physics remains to be understood by further analysis of our model.

At the moment we are interested in the renormalization properties of (6.5), first of all in the cancellation of the quadratic divergences. Let us comment that it is possible to choose a specific type of deformation, such that it leads to $K^{ab}K_{ab} = K^{*ab}K_{ab}^* = 0$. This choice takes the H^3 term in (5.21) to zero and simplifies calculations drastically. More important, renormalization properties of our model might turn out to be better with this choice.

One should analyze microcausality of our theory since a non-local interaction term appears in the action. Also, the construction of gauge theories on this deformed superspace is planned for future research.

Concerning different types of deformation, we also analyzed a model with $\mathcal{F} = e^{\frac{1}{2}C^{\alpha\beta}D_\alpha \otimes D_\beta}$ which leads to the deformation discussed in [12]. Comments on this work are planned for the next publication.

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Final remark This work was completed in July (all except the section 7, for which only the idea was given at that time) when all the authors enjoyed the hospitality of II Institute for Theoretical Physics and DESY, Hamburg. Sadly and unexpectedly Julius Wess passed away in August. We were left with piles of handwritten calculations and no paper written. Much more important, we stayed without Julius's enormous knowledge and experience, his ideas, encouragement, support and his friendship. He enjoyed this work very much and we only hope that he would not object the way we wrote it up too much.

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